# A Lambda Calculus for Transfinite Arrays 

Unifying Arrays and Streams

ARTJOMS ŠINKAROVS, Heriot-Watt University SVEN-BODO SCHOLZ, Heriot-Watt University


#### Abstract

Array programming languages allow for concise and generic formulations of numerical algorithms, thereby providing a huge potential for program optimisation such as fusion, parallelisation, etc. One of the restrictions that these languages typically have is that the number of elements in every array has to be finite. This means that implementing streaming algorithms in such languages requires new types of data structures, with operations that are not immediately compatible with existing array operations or compiler optimisations.

In this paper, we propose a design for a functional language that natively supports infinite arrays. We use ordinal numbers to introduce the notion of infinity in shapes and indices. By doing so, we obtain a calculus that naturally extends existing array calculi and, at the same time, allows for recursive specifications as they are found in stream- and list-based settings. Furthermore, the main language construct that can be thought of as an $n$-fold cons operator gives rise to expressing transfinite recursion in data, something that lists or streams usually do not support. This makes it possible to treat the proposed calculus as a unifying theory of arrays, lists and streams. We give an operational semantics of the proposed language, discuss design choices that we have made, and demonstrate its expressibility with several examples. We also demonstrate that the proposed formalism preserves a number of well-known universal equalities from array/list/stream theories, and discuss implementation-related challenges.


## CCS Concepts: • Theory of computation $\rightarrow$ Operational semantics;

Additional Key Words and Phrases: ordinals, arrays, semantics, functional languages
ACM Reference format:
Artjoms Šinkarovs and Sven-Bodo Scholz. 2017. A Lambda Calculus for Transfinite Arrays. Proc. ACM Program. Lang. 1, 1, Article 1 (January 2017), 28 pages.
https://doi.org/10.1145/nnnnnnn.nnnnnnn

## 1 INTRODUCTION

Array-based computation offers many appealing properties when dealing with large amounts of homogeneous data. All data can be accessed in $O(1)$ time, storage is compact, and array programs typically lend themselves to data-parallel execution. Another benefit arises from the fact that many applications naturally deal with data that is structured along several independent axes of linearly ordered elements.

Besides these immediate benefits, the structuring facilities of arrays offer significant opportunities for developing elaborate array calculi, such as Mullin's $\psi$-calculus [47, 48], Nial [21] and the many APL-inspired array languages $[8,12,30]$. These calculi benefit programmers in several ways.

Firstly, they improve programmer productivity. Array calculi provide the grounds for a rich set of generic operators. Having these readily available, programmers can compose programs more quickly; the resulting algorithms are typically more versatile than implementations that manipulate arrays directly on an element-by-element basis.

Secondly, array calculi help improving program correctness. The operators that build the foundation of any given array calculus come with a wealth of properties that manifest in equalities that hold for all arrays. With these properties available, programming becomes less error-prone, e.g.
avoidance of out-of-bound accesses through the use of array-oriented operators. It also becomes easier to reason formally about programs and their behaviour.

Finally, array calculi help compilers to optimise and parallelise programs. The aforementioned equalities can also be leveraged when it comes to high-performance parallel executions. They allow compilers to restructure both algorithms and data structures, enabling improvements such as better data locality [11, 22], better vectorisation [3,53,57], and streaming through accelerator devices [2, 24].

These advantages have inspired many languages and their attendant tool chains including various APL implementations [8, 17, 30, 31], parallel arrays in Haskell [38], push-pull arrays in Haskell [51], and arrays in SAC [23] and Futhark [28].

Despite generic specification and the ability to stream finite arrays, array languages typically cannot deal with infinite streams. If an existing application for finite arrays needs to be extended to deal with infinite streams, a complete code rewrite is often required, even if the algorithmic pattern applied to the array elements is unchanged. Retrofitting such streaming often obfuscates the core algorithm, and adds overhead when the algorithm is applied to finite streams. If the overhead is non-negligible, both code versions need to be maintained and, if the finiteness of the data is not known a priori, a dynamic switch between them is required.

Streaming style leads to a completely different way of thinking about data. Very similar to programming on lists, traditional streaming deals with individual recursive acts of creation or consumption. This is appealing, as elegant recursive data definitions become possible and element insertion and deletion can be implemented efficiently, which is difficult to achieve in a traditional array setting. Another aspect of streams is the inevitable temporal existence of parts of streams or lists, which quite well matches the lazy regime prevalent in list-based languages, but is usually at odds with obtaining high-performance parallel array processing.

In this paper, we try to tackle this limitation of array languages when it comes to infinite structures. Specifically, we look at extending array languages in a consistent way, to support streaming through infinite dimensions. Our aim is to avoid switching to a traditional streaming approach, and to stay within the array paradigm, thereby making it possible to use the same algorithm specification for both finite and infinite inputs, possibly maintaining the benefits of a given underlying array calculus. We also hope that excellent parallel performance can be maintained for the finite cases, and that typical array-based optimisations can be applied to both the finite and infinite cases.

We start from an applied $\lambda$-calculus supporting finite $n$-dimensional arrays, and investigate extensions to support infinite arrays. The design of this calculus aims to provide a solid basis for several array calculi, and to facilitate compilation to high-performance parallel code.

In extending the calculus to deal with infinities, we pay particular attention to the algebraic properties that are present in the finite case, and how they translate into the infinite scenario. The key insight here is that when ordinals are used to describe shapes and indices of arrays, many useful properties can be preserved. Borrowing from the nomenclature of ordinals, we refer to these ordinal-indexed infinite arrays as transfinite arrays. We identify and minimize requisite semantic extensions and modifications. We also look into the relationship between the resulting array-based $\lambda$-calculus and classical streaming. Finally, we look at several examples, and discuss implementation issues.

The individual contributions of this paper are as follows:
(1) We define an applied $\lambda$-calculus on finite arrays, and its operational semantics. The calculus is a rather generic core language that implicitly supports several array calculi as well as compilation to highly efficient parallel code.
(2) We expand the $\lambda$-calculus to support infinite arrays and show that the use of ordinals as indices and shapes creates a wide range of universal equalities that apply to finite and transfinite arrays alike.
(3) We show that the proposed calculus also maintains many streaming properties even in the context of transfinite streaming.
(4) We show that the proposed calculus inherently supports transfinite recursion. Several examples are contrasted to traditional list-based solutions.
(5) We describe a prototypical implementation ${ }^{1}$, and briefly discuss the opportunities and challenges involved.
We start with a description of the finite array calculus and naive extensions for infinite arrays in Section 2, before presenting the ordinal-based approach and its potential in Sections 3-5. Section 6 presents our prototypical implementation. Related work is discussed in Section 7; we conclude in Section 8.

## 2 EXTENDING ARRAYS TO INFINITY

We define an idealised, data-parallel array language, based on an applied $\lambda$-calculus that we call $\lambda_{\alpha}$. The key aspect of $\lambda_{\alpha}$ is built-in support for shape- and rank-polymorphic array operations, similar to what is available in APL [32], J [35], or SAC [23].

In the array programming community, it is well-known $[19,34]$ that basic design choices made in a language have an impact on the array algebras to which the language adheres. While we believe that our proposed approach is applicable within various array algebras, we chose one concrete setting for the context of this paper. We follow the design decisions of the functional array language SAC , which are compatible with many array languages, and which were taken directly from K.E. Iverson's design of APL.

DD 1 All expressions in $\lambda_{\alpha}$ are arrays. Each array has a shape which defines how components within arrays can be selected.
DD 2 Scalar expressions, such as constants or functions, are 0-dimensional objects with empty shape. Note that this maintains the property that all arrays consist of as many elements as the product of their shape, since the product of an empty shape is defined through the neutral element of multiplication, i.e. the number 1.
DD 3 Arrays are rectangular - the index space of every array forms a hyper-rectangle. This allows the shape of an array to be defined by a single vector containing the element count for each axis of the given array.
DD 4 Nested arrays that cater for inhomogeneous nesting are not supported. Homogeneously nested array expressions are considered isomorphic with non-nested higher-dimensional arrays. Inhomogeneous nesting, in principle, can be supported by adding dual constructs for enclosing and disclosing an entire array into a singleton, and vice versa. DD 2 implies that functions and function application can be used for this purpose.
DD $5 \lambda_{\alpha}$ supports infinitely many distinct empty arrays that differ only in their shapes. In the definition of array calculi, the choice whether there is only one empty array or several has consequences on the universal equalities that hold. While a single empty array benefits valuefocussed equalities, structural equalities require knowledge of array shapes, even when those arrays are empty. In this work, we assume an infinite number of empty arrays; any array with at least one shape element being 0 is empty. Empty arrays with different shape are considered distinct. For example, the empty arrays of shape $[3,0]$ and $[0]$ are different arrays.

[^0]
### 2.1 Syntax Definition and Informal Semantics of $\lambda_{\alpha}$



Fig. 1. The syntax of $\lambda_{\alpha}$
We define the syntax of $\lambda_{\alpha}$ in Fig. 1. Its core is an untyped, applied $\lambda$-calculus. Besides scalar constants, variables, abstractions and applications, we introduce conditionals, a recursive let operator and some basic functions on the constants, including arithmetic operations such as $+,-, *, /$, a remainder operation denoted as $\%$, and comparisons $<,<=,=$, etc. The actual support for arrays as envisioned by the aforementioned design principles is provided through five further constructs: array construction, selection, shape operation, reduce and imap combinators.

All arrays in $\lambda_{\alpha}$ are immutable. Arrays can be constructed by using potentially nested sequences of scalars in square brackets. For example, [1, 2, 3, 4] denotes a four-element vector, while [[1, 2], [3, 4]] denotes a two-by-two-element matrix. We require any such nesting to be homogeneous, for adherence to DD 4. For example, the term [[1, 2], [3]] is irreducible, so does not constitute a value.

The dual of array construction is a built-in operation for element selection, denoted by a dot symbol, used as an infix binary operator between an array to select from, and a valid index into that array. A valid index is a vector containing as many elements as the array has dimensions; otherwise it is undefined.

$$
[1,2,3,4] \cdot[0]=1 \quad[[1,2],[3,4]] \cdot[1,1]=4 \quad[[1,2],[3,4]] \cdot[1]=\perp
$$

The third array-specific addition to $\lambda_{\alpha}$ is the primitive shape operation, denoted by enclosing vertical bars. It is applicable to arbitrary expressions, as demanded by DD 1 , and it returns the shape of its argument as a vector, leveraging DD 3. For our running examples, we obtain: $|[1,2,3,4]|=$ $[4]$ and $|[[1,2],[3,4]]|=[2,2]$. DD 5 and DD 2 imply that we have:

$$
|[]|=[0] \quad|[[]]|=[1,0] \quad \mid \text { true }|=[] \quad| 42|=[] \quad| \lambda x \cdot x \mid=[]
$$

$\lambda_{\alpha}$ includes a reduce combinator which in essence, it is a variant of foldl, extended to allow for multi-dimensional arrays instead of lists. reduce takes three arguments: the binary function, the neutral element and the array to reduce. For example, we have:

$$
\text { reduce }(+) 0[[1,2],[3,4]]=((((0+1)+2)+3)+4)
$$

assuming row-major traversal order. This allows for shape-polymorphic reductions such as:

```
sum \(\equiv \lambda \mathrm{a}\). reduce \((\lambda \mathrm{x} \cdot \lambda \mathrm{y} \cdot \mathrm{x}+\mathrm{y}) 0 \mathrm{a}\); also works for scalars and empty arrays
```

The final, and most elaborate, language construct is the imap (index map) construct. It bears some similarity to the classical map operation, but instead of mapping a function over the elements of an array, it constructs an array by mapping a function over all legal indices into the index space denoted by a given shape expression ${ }^{2}$. Added flexibility is obtained by supporting a piecewise definition of the function to be mapped. Syntactically, the imap-construct starts out with the keyword imap, followed by a description of the result shape (rule $s$ in Fig. 1). The shape description is followed by a curly bracket that precedes the definition of the mapping function. This function can be defined piecewise by providing a set of index-range expression pairs. We demand that the set of index ranges constitutes a partitioning of the overall index space defined through the result shape expression, i.e. their union covers the entire index space and the index ranges are mutually disjoint. We refer to such index ranges as generators (rule $g$ in Fig. 1), and we call a pair of a generator and its subsequent expression a partition. Each generator defines an index set and a variable (denoted by $x$ in rule $g$ in Fig. 1) which serves as the formal parameter of the function to be mapped over the index set. Generators can be defined in two ways: by means of two expressions which must evaluate to vectors of the same shape, constituting the lower and upper bounds of the index set, or by using the underscore notation which is syntactic sugar for the following expansion rule:
$(\operatorname{imap} \mathrm{s}\left\{\__{-}(\mathrm{iv}) \ldots\right) \equiv(\operatorname{imap} \mathrm{s}\{[\underbrace{0, \ldots, 0}]<=\mathrm{iv}<\mathrm{s}: \ldots)$
$n$
assuming that $|s|=[n]$. The variable name of a generator can be referred to in the expression of the corresponding partition.

The <= and < operators in the generators can be seen as element-by-element array counterparts of the corresponding scalar operators which, jointly, specify sets of constraints on the indices described by the generators. As the index-bounds are vectors, we have:

$$
v_{1}<=v_{2} \Longrightarrow\left|v_{1}\right| \cdot[0]=\left|v_{2}\right| \cdot[0] \wedge \forall 0<=i<\left|v_{1}\right| \cdot[0]: v_{1} \cdot[i]<=v_{2} \cdot[i]
$$

In the rest of the paper, we use the same element-wise extensions for scalar operators, denoting the non-scalar versions with dot on top: $c=a \dot{+} b \Longrightarrow c . i=a . i+b . i$. This often helps to simplify the notation ${ }^{3}$.

As an example of an imap, consider an element-wise increment of an array $a$ of shape [ $n$ ]. While a classical map-based definition can be expressed as map $(\lambda x . x+1) a$, using imap, the same operation can be defined as:

```
imap [n] { [0] <= iv < [n]: a.iv + 1
```

Having mapping functions from indices to values rather than values to values adds to the flexibility of the construct. Arrays can be constructed from shape expressions without requiring an array of the same shape available:

```
imap [3,3] { [0,0] <= iv < [3,3]: iv.[0]*3 + iv.[1]
```

defines a 2 -dimensional array $[[0,1,2],[3,4,5],[6,7,8]]$. Structural manipulations can be defined conveniently as well. Consider a reverse function, defined as follows:

```
reverse \equiv \lambdaa.imap |a| { [0] <= iv < |a|: a.(|a|\doteqiv-[1])
```

[^1]In order to express this with map, one needs to construct an intermediate array, where indices of $a$ appear as values. Note also that the explicit shape of the imap construct makes it possible to define shape-polymorphic functions in a way similar to our definition of reverse. An element-wise increment for arbitrarily shaped arrays can be defined as:
increment $\equiv \lambda \mathrm{a} . \operatorname{imap}|\mathrm{a}|\left\{\_(\mathrm{iv}): \mathrm{a} \cdot \mathrm{iv}+1\right.$; also works for scalars \& empty arrays
DD 4 allows imap to be used for expressing operations in terms of $n$-dimensional sub-structures. All that is required for this is that the expressions on the right hand side of all partitions evaluate to non-scalar values. For example, matrices can be constructed from vectors. Consider the following expression:

```
imap [n] {[0]<= iv < [n]: [1,2,3,4] ; non-scalar partitions (incorrect attempt)
```

Its shape is $[n, 4]$; however, this shape no longer can be computed without knowing the shape of at least one element. If the overall result array is empty, its shape determination is a non-trivial problem. To avoid this situation, we require the programmer to specify the result shape by means of two shape expressions separated by a vertical bar: see the rule (generic imap) in Fig. 1. We refer to these two shape expressions as the frame shape which specifies the overall index range of the imap construct as well as the cell shape which defines the shape of all expressions at any given index. The concatenation of those two shapes is the overall shape of the resulting array. For more discussions related to the concepts of frame and cell shapes, see [6, 7, 9]. The above imap expression therefore needs to be written as:
$\operatorname{imap}[\mathrm{n}] \mid[4]\{[0]<=\mathrm{iv}<[\mathrm{n}]:[1,2,3,4]$; non-scalar partitions (correct)
to be a legitimate expression of $\lambda_{\alpha}$. The (scalar imap) case in Fig. 1, which we use predominantly in the paper, can be seen as syntactic sugar for the generic version, with the second expression being an empty vector.

### 2.2 Formal Semantics of $\lambda_{\alpha}$

In this section, we offer a brief overview of the semantics. A complete semantics can be found in [58].

In $\lambda_{\alpha}$, evaluated arrays are pairs of shape and element tuples. A shape tuple consists of numbers, and an element tuple consists of numbers, booleans or functions closures. We denote pairs and tuples, as well as element selection and concatenation on them, using the following notation:

$$
\vec{a}=\left\langle a_{1}, \ldots, a_{n}\right\rangle \Longrightarrow \vec{a}_{i}=a_{i} \quad\left\langle a_{1}, \ldots, a_{n}\right\rangle++\left\langle b_{1}, \ldots, b_{m}\right\rangle=\left\langle a_{1}, \ldots, a_{n}, b_{1}, \ldots, b_{m}\right\rangle
$$

To denote the product of a tuple of numbers, we use the following notation:

$$
\vec{s}=\left\langle s_{1}, \ldots, s_{n}\right\rangle \Longrightarrow \otimes \vec{s}=s_{n} \cdots s_{1} \cdot 1
$$

When a tuple is empty, its product is one. An array is rectangular, so its shape vector specifies the extent of each axis. The number of elements of each array is finite. Element vectors contain all the elements in a linearised form. While the reader can assume row-major order, formally, it suffices that a fixed linearisation function $F_{\vec{s}}$ exists which, given a shape vector $\vec{s}=\left\langle s_{1}, \ldots, s_{n}\right\rangle$, is a bijection between indices $\left\{\langle 0, \ldots, 0\rangle, \ldots,\left\langle s_{1}-1, \ldots, s_{n}-1\right\rangle\right\}$ and offsets of the element vector: $\{1, \ldots, \otimes \vec{s}\}$. Consider, as an example, the array [[1, 2], [3, 4]], with $F$ being row-major order. This array is evaluated into the shape-tuple element-tuple pair $\langle\langle 2,2\rangle,\langle 1,2,3,4\rangle\rangle$. Scalar constants are arrays with empty shapes. We have 5 evaluating to $\langle\rangle,\langle 5\rangle\rangle$. The same holds for booleans and function closures: true evaluates to $\langle\rangle,\langle$ true $\rangle\rangle$ and $\lambda x . e$ evaluates to $\langle\rangle,\langle\llbracket \lambda x . e, \rho \rrbracket\rangle\rangle$.
$F$ is an invariant to the presented semantics. In finite cases, the usual choices of $F$ are row-major order or column-major order. In infinite cases, this might be not the best option, and one could consider space-filling curves instead. $F$ is only relevant for two operations; the creation of array
values and the selection of elements from it. Selections relate the indices of the index vectors to the axes of the arrays following the order of nesting and starting with the index 0 on each level. We have: $[[1,2],[3,4]][1,0]=3$, Assuming $F$ is row-major, $F_{\langle 2,2\rangle}(\langle 1,0\rangle)$ equals 2 which, when used as index into $\langle\langle 2,2\rangle,\langle 1,2,3,4\rangle\rangle$ returns the intended result 3.

The inverse of $F$ is denoted as $F_{\vec{s}}^{-1}$ and for every legal offset $\{1, \ldots, \otimes \vec{s}\}$ it returns an index vector for that offset.

Deduction rules. To define the operational semantics of $\lambda_{\alpha}$, we use a natural semantics, similar to the one described in [36]. To make sharing more visible, instead of a single environment $\rho$ that maps names to values, we introduce a concept of storage; environments map names to pointers and storage maps pointers to values. Environments are denoted by $\rho$ and are ordered lists of namepointer pairs. Storage is denoted by $S$ and consists of an ordered list of pointer-value pairs.

Formally, we construct storage and environments as lists of pointer-value and variable-pointer bindings, respectively, using comma to denote extensions:

$$
S::=\emptyset|S, p \mapsto v \quad \rho::=\emptyset| \rho, x \mapsto p
$$

A look-up of a storage or an environment is performed right to left and is denoted as $S(p)$ and $\rho(x)$, respectively. Extensions are denoted with comma. Semantic judgements can take two forms:

$$
S ; \rho \vdash e \Downarrow S^{\prime} ; p \quad S ; \rho \vdash e \Downarrow S^{\prime} ; p \Rightarrow v
$$

where $S$ and $\rho$ are initial storage and environment and $e$ is a $\lambda_{\alpha}$ expression to be evaluated. The result of this evaluation ends up in the storage $S^{\prime}$ and the pointer $p$ points to it. The latter form of a judgement is a shortcut for: $S ; \rho \vdash e \Downarrow S^{\prime} ; p \wedge S^{\prime}(p)=v$.

Values. The values in this semantics are constants (including arrays) and $\lambda$-closures which contain the $\lambda$ term and the environment where this term shall be evaluated:

$$
\langle\langle\ldots\rangle,\langle\ldots\rangle\rangle \quad\langle\rangle,\langle\llbracket \lambda x . e, \rho \rrbracket\rangle\rangle
$$

Meta-operators. Further in this section we use the following meta-operators:
$\mathbf{E}(v)$ Lift the internal representation of a vector or a number into a valid $\lambda_{\alpha}$ expression. For example: $\mathbf{E}(5)=5, \mathbf{E}(\langle 1,2,3\rangle)=[1,2,3]$, etc.
$\left\langle\vec{s}, \_\right\rangle$We use underscore to omit the part of a data structure, when binding names. For example: $S ; p \Rightarrow\left\langle\vec{s}, \_\right\rangle$refers to binding the variable $\vec{s}$ to the shape of $S(p)$ which must be a constant.

### 2.3 Core Rules

In $\lambda_{\alpha}$, the rules for the $\lambda$-calculus core, i.e. constants, variables, abstractions and applications are straightforward adaptations of the standard rules for strict functional languages to our notation with storage and pointers:

| Const-Scal |  |
| :--- | :--- |
| $\frac{c}{}$ is scalar | Var <br> $; \rho \vdash c \Downarrow S_{1}, p \mapsto\langle\langle \rangle,\langle c\rangle\rangle ; p$ | | $x \in \rho$ |
| :--- |
| $S ; \rho \vdash x \Downarrow S ; \rho(x)$ |


|  |  |
| :--- | :---: |
| ABS | App <br> $S ; \rho \vdash \lambda x . e \Downarrow S, p \mapsto\langle\langle \rangle,\langle\llbracket \lambda x . e, \rho \rrbracket\rangle\rangle ; p$ | | $S ; e_{1} \Downarrow S_{1} ; p_{1} \Rightarrow\left\langle\langle \rangle, \llbracket \lambda x . e, \rho_{1} \rrbracket\right\rangle$ |
| :---: |
| $S_{1} ; \rho \vdash e_{2} \Downarrow S_{2} ; p_{2} \quad S_{2} ; \rho_{1}, x \mapsto p_{2} \vdash e \Downarrow S_{3} ; p_{3}$ |
| $S ; \rho \vdash e_{1} e_{2} \Downarrow S_{3} ; p_{3}$ |

As an illustration, consider the evaluation of $(\lambda x . x) 42$ :

| $\emptyset ; \emptyset$ | $(\lambda x . x) 42$ | Abs |
| :--- | ---: | :--- |
| $S_{1}=p_{1} \mapsto\langle\langle \rangle, \llbracket \lambda x \cdot x, \emptyset \rrbracket\rangle ; \emptyset$ | $p_{1} 42$ | Const-ScAL |
| $S_{2}=S_{1}, p_{2} \mapsto\langle\langle \rangle,\langle 42\rangle\rangle ; \emptyset$ | $p_{1} p_{2}$ | App |
| $S_{2} ; x \mapsto p_{2}$ | $x$ | VAR |
| $S_{2} ; \emptyset$ | $p_{2}$ | $\square$ |

We start with an empty storage and an empty environment. The outer application demands that the App-rule be used. It enforces three computations: the evaluation of the function, the evaluation of the argument and the evaluation of the function body with an appropriately expanded environment. The function is evaluated by the Abs-rule which adds a closure $p_{1} \mapsto\langle\langle \rangle, \llbracket \lambda x . x, \emptyset \rrbracket\rangle$ to the storage and returns the pointer $p_{1}$ to it. The argument is evaluated by the Const-Scal-rule which adds $p_{2} \mapsto\langle\langle \rangle,\langle 42\rangle\rangle$ to the storage and returns $p_{2}$. Finally, the App-rule demands the evaluation of the body of the function with an environment $\rho_{1}=x \mapsto p_{2}$. The body being just the variable $x$, the Var-rule gives us $S_{2} ; p_{2}$ as final result.

The rules for array constructors and array selections are rather straightforward as well. Both these constructs are strict:

Imm-Array

$$
\begin{gathered}
n \geq 1 \quad \stackrel{V_{i=1}^{\forall} S_{i} ; \rho \vdash c_{i} \Downarrow S_{i+1} ; p_{i}}{ } \begin{array}{c}
n=\left\langle p_{1}, \ldots, p_{n}\right\rangle \begin{array}{c}
\text { AllSameShape }\left(S_{n+1}, P\right)
\end{array} \quad S^{\prime}=S_{n+1}, p_{o} \mapsto\langle\langle 1\rangle,\langle n\rangle\rangle, p_{i} \mapsto S_{n+1}\left(p_{1}\right) \\
S^{\prime}, \rho \vdash i m a p_{1} p_{o} \mid p_{i}\left\{\langle i-1\rangle \mapsto p_{i} \mid i \in\{1, \ldots, n\}\right\} \Downarrow S^{\prime \prime} ; p
\end{array} \\
\hline S_{1} ; \rho \vdash\left[c_{1}, \ldots, c_{n}\right] \Downarrow S^{\prime \prime} ; p
\end{gathered}
$$

Imm-Array-empty

$$
\overline{S ; \rho \vdash[] \Downarrow S, p \mapsto\langle\langle 0\rangle,\langle \rangle\rangle ; p}
$$

$$
\begin{aligned}
& \text { Sel-strict } \\
& \frac{S ; \rho \vdash i \Downarrow S_{1} ; p_{i} \Rightarrow\langle\langle d\rangle, \vec{\imath}\rangle \quad S_{1} ; \rho \vdash a \Downarrow S_{2} ; p_{a} \Rightarrow\langle\vec{s}, \vec{a}\rangle \quad k=F_{\vec{s}}(\vec{\imath})}{S ; \rho \vdash a . i \Downarrow S_{3}, p \mapsto\left\langle\langle \rangle,\left\langle\vec{a}_{k}\right\rangle\right\rangle ; p}
\end{aligned}
$$

Empty arrays are put into the storage with shape [0] (Imm-Array-empty-rule). Non-empty arrays (Imm-Array-rule) evaluate all the components and ensure that they are all of the same finite shape. Subsequently, we assemble evaluated components into the resulting array value ensuring that the flattening adheres to $F$. This is achieved by using an auxiliary term imap ${ }_{1}$. It takes the form $\operatorname{imap}_{1} p_{o} \mid p_{i}\left\{\vec{\imath}^{1} \mapsto p_{\vec{l}^{1}}, \ldots, \vec{\imath}^{n} \mapsto p_{i^{n}}\right\}$ where $p_{o}$ and $p_{i}$ are pointers to frame and cell shapes, and the set $\left\{\vec{\imath}^{1} \mapsto p_{i^{1}}, \ldots, \vec{\imath}^{n} \mapsto p_{i^{n}}\right\}$ contains pairs of frame-shape indices and value pointers for all legal indices into the frame shape. The formal definition of the deduction rule for imap ${ }_{1}$ is provided in [58, Sec 2.1.1].

The rule for selection (Sel-strict-rule) first evaluates the array we are selecting from, and the index vector specifying the array index we wish to select. Then, we compute the offset into the data vector by applying $F$ to the index vector. Finally, we get the scalar value at the corresponding index. When applying $F$, we implicitly check that:

- the index is within bounds $1 \leq k \leq \otimes \vec{s}$, as $F_{\vec{s}}$ is undefined outside the index space bounded by $\vec{s}$; and
- the index vector and the shape vector are of the same length, which means that selections evaluate scalars and not array sub-regions.

IMap. In order to keep the imap rule reasonably concise, we introduce two separate rules, a rule GEn for evaluating the generator bounds, and the main rule for imap, the Imap-Strict-Rule:

IMAP-Strict

$$
\begin{aligned}
& S ; \rho \vdash e_{\text {out }} \Downarrow S_{1} ; p_{\text {out }} \Rightarrow\left\langle\left\langle d_{o}\right\rangle, \overrightarrow{s_{\text {out }}}\right\rangle \quad S_{1} ; \rho \vdash e_{\text {in }} \Downarrow S_{2} ; p_{\text {in }} \Rightarrow\left\langle\left\langle d_{i}\right\rangle, \overrightarrow{s_{\text {in }}}\right\rangle \\
& \hat{S}_{1}=S_{2} \quad \underset{i=1}{\forall} \hat{S}_{i} ; \rho \vdash g_{i} \Downarrow \hat{S}_{i+1} ; p_{g_{i}} \Rightarrow \bar{g}_{i} \quad \text { FormsPartition }\left(s_{\text {out }},\left\{\bar{g}_{1}, \ldots, \bar{g}_{n}\right\}\right) \\
& \bar{S}_{1}=\hat{S}_{n+1} \quad \forall(i, \vec{\imath}) \in \operatorname{Enumerate}\left(s_{\text {out }}\right) \exists k: \begin{array}{l}
\vec{\imath} \in \bar{g}_{k} \wedge \bar{g}_{k}=\operatorname{Gen}\left(x_{k},{ }_{-},{ }_{-}\right) \\
\bar{S}_{i}, p \mapsto\left\langle\left\langle d_{o}\right\rangle, \vec{\imath}\right\rangle ; \rho, x_{k} \mapsto p \vdash e_{k} \Downarrow \bar{S}_{i}^{\prime} ; p_{\vec{i}} \\
\bar{S}_{i}^{\prime} ; \rho x
\end{array} \\
& \bar{S}_{i}^{\prime} ; \rho, x \mapsto p_{\vec{\imath}} \vdash|x| \Downarrow \bar{S}_{i+1} ; p_{\vec{\imath}}^{\prime} \Rightarrow\left\langle\left\langle d_{i}\right\rangle, \overrightarrow{s_{\mathrm{in}}}\right\rangle \\
& \bar{S}_{\otimes s_{\text {out }}+1}, \rho \vdash \operatorname{imap}_{1} p_{\text {out }} \mid p_{\text {in }}\left\{\vec{\imath} \mapsto p_{\imath} \mid\left(\_, \vec{\imath}\right) \in \operatorname{Enumerate}\left(s_{\text {out }}\right)\right\} \Downarrow S^{\prime} ; p \\
& S ; \rho \vdash \operatorname{imap} e_{\text {out }} \left\lvert\, e_{\text {in }}\left\{\begin{array}{ll}
g_{1}: & e_{1}, \\
\cdots & \\
g_{n}: & e_{n}
\end{array} \quad \Downarrow S^{\prime} ; p\right.\right. \\
& \text { Gen } \\
& \frac{S ; \rho \vdash e_{1} \Downarrow S_{1} ; p_{1} \Rightarrow\langle\langle n\rangle, \vec{l}\rangle \quad S_{1} ; \rho \vdash e_{2} \Downarrow S_{2} ; p_{2} \Rightarrow\langle\langle n\rangle, \vec{u}\rangle}{S ; \rho \vdash\left(e_{1} \leq x<e_{2}\right) \Downarrow S, p \mapsto \operatorname{Gen}(x, \vec{l}, \vec{u}) ; p}
\end{aligned}
$$

The GEN-rule introduces auxiliary values $\operatorname{Gen}(x, \vec{l}, \vec{u})$ which are triplets that keep a variable name, lower bound and upper bound of a generator together. These auxiliary values are references only by the rule for imap.

Evaluation of an imap happens in three steps. First, we compute shapes and generators, making sure that generators form a partition of $s_{\text {out }}^{\overrightarrow{0}}$ (FormsPartition is responsible for this). Secondly, for every valid index defined by the frame shape (Enumerate generates a set of offset-index-vector pairs), we find a generator that includes the given index (denoted $\vec{\imath} \in \bar{g}_{k}$ ). We evaluate the generator expression $e_{k}$, binding the generator variable $x_{k}$ to the corresponding index value and check that the result has the same shape as $p_{\mathrm{in}}$. Finally, we combine evaluated expressions for every index of the frame shape into $i m a p_{1}$ for further extraction of scalar values.

All missing rules, including built-in operations, conditionals and recursion through the letrecconstruct are straightforward adaptations of the standard rules. They can be found in [58]. Formal definitions of helper functions, such as AllSameShape, will also be found there.

### 2.4 Infinite Arrays

In order to support infinite arrays, we introduce the notion of infinity in $\lambda_{\alpha}$, and we allow infinities to appear in shape components. Syntactically, this can be achieved by adding a symbol for infinity, as shown in Fig. 2. For disambiguation, we refer to the extended version of $\lambda_{\alpha}$ as $\lambda_{\alpha}^{\infty}$. Adding $\infty$


Fig. 2. The syntax of $\lambda_{\alpha}^{\infty}$
has several implications. First of all, our built-in arithmetic needs to be extended. We treat infinity in the usual way, applying the model commonly known as a Riemann sphere. That is:

$$
z+\infty=\infty \quad z \times \infty=\infty \quad \frac{z}{\infty}=0 \quad \frac{z}{0}=\infty
$$

The following operations are undefined:

$$
\infty+\infty \quad \infty-\infty \quad \infty \times 0 \quad \frac{0}{0} \quad \frac{\infty}{\infty}
$$

While these additions to the semantics are trivial, allowing infinity to appear in shapes has a more profound impact on our semantics. Our rule for imap-constructs (ImAp-Strict) forces the evaluation of all elements. If our result shape contains infinity, this can no longer be done. As we want to maintain a strict evaluation regime for function applications in general, we turn our imap-construct into a lazy data-structure which does not immediately compute its elements, but only does so when individual elements are being inspected. For this purpose, we extend our set of allowed values of our semantics with an imap-closure:

$$
\| i m a p p_{\text {out }} \left\lvert\, p_{\text {in }}\left\{\begin{array}{ll}
\bar{g}_{1}: & e_{1}, \\
\ldots & , \rho \\
\bar{g}_{n}: & e_{n}
\end{array} \|\right.\right.
$$

The imap closure contains pointers to frame and element shapes ( $p_{\text {out }}$ and $p_{\text {in }}$ correspondingly), the list of partitions, where generators have been evaluated and the environment in which the imap shall be evaluated. The overall idea is to update, in place, this closure whenever individual elements are computed. With this extension, we can now replace our strict imap-rule by a lazy variant:

$$
\begin{aligned}
& \text { IMAP-LAZY } \\
& S ; \rho \vdash e_{\text {out }} \Downarrow S_{1} ; p_{\text {out }} \Rightarrow\left\langle\left\langle \_\right\rangle, s_{\text {out }}\right\rangle \quad S_{1} ; \rho \vdash e_{\text {in }} \Downarrow S_{2} ; p_{\text {in }} \Rightarrow\left\langle\left\langle_{-}\right\rangle,{ }_{-}\right\rangle \\
& \xrightarrow[\hat{S}_{1}=S_{2} \quad \stackrel{n}{\forall} \hat{S}_{i=1} ; \rho \vdash g_{i} \Downarrow \hat{S}_{i+1} ; p_{g_{i}} \Rightarrow \bar{g}_{i} \quad \text { FormsPartition }\left(\overrightarrow{s_{\text {out }}},\left\{\bar{g}_{1}, \ldots, \bar{g}_{n}\right\},\right)]{ } \\
& S ; \rho \vdash \operatorname{imap} e_{\text {out }} \left\lvert\, e_{\text {in }}\left\{\begin{array}{ll}
g_{1}: & e_{1}, \\
\ldots & \quad \\
g_{n}: & e_{n}
\end{array} \quad \Downarrow \hat{S}_{n+1}, p \mapsto \| \text { imap } p_{\text {out }} \left\lvert\, p_{\text {in }}\left\{\begin{array}{ll}
\bar{g}_{1}: & e_{1}, \\
\ldots & ; \rho \\
\bar{g}_{n}: & e_{n}
\end{array}\right]\right. ; p\right.\right.
\end{aligned}
$$

We can see that the new rule for imap-constructs, in essence, performs a subset of what the strict rule from the previous section does. It still forces the result shapes, it still computes the boundaries of the generators, and it checks the validity of the overall generator set. Once these computations have been done, further element computation is delayed and an imap-closure is created instead.

The actual computation of elements is triggered upon element selection. Consequently, we need a second selection rule which can deal with imap closures in the array argument position:

$$
\begin{aligned}
& \text { Sel-lazy-imap } \\
& S ; \rho \vdash i \Downarrow S_{1} ; p_{i} \Rightarrow\left\langle\left\langle \_\right\rangle, \vec{v}\right\rangle \quad S_{1} ; \rho \vdash a \Downarrow S_{2} ; p_{a} \Rightarrow \llbracket i m a p p_{\mathrm{out}} \left\lvert\, p_{\mathrm{in}}\left\{\begin{array}{ll}
\bar{g}_{1} & e_{1} \\
\ldots & , \rho^{\prime} \\
\bar{g}_{n} & e_{n}
\end{array}\right]\right. \\
& S_{2}\left(p_{\text {out }}\right)=\left\langle\langle m\rangle, \_\right\rangle \quad(\vec{\imath}, \vec{\jmath})=\operatorname{Split}(m, \vec{v}) \\
& \exists k: \vec{\imath} \in \bar{g}_{k} \quad \bar{g}_{k}=\operatorname{Gen}\left(x_{k},{ }_{-},{ }_{-}\right) \quad S_{2}, p \mapsto \mathbf{E}(\vec{\imath}) ; \rho^{\prime}, x_{k} \mapsto p \vdash e_{k} \Downarrow S_{3} ; p_{\vec{\imath}} \\
& S_{3} ; \rho^{\prime}, x \mapsto p_{\vec{\imath}} \vdash x \cdot \mathbf{E}(\vec{\jmath}) \Downarrow S_{4} ; p \quad S_{5}=\operatorname{UpdateIMap}\left(S_{4}, p_{a}, \vec{\imath}, p_{\vec{\imath}}\right) \\
& S ; \rho \vdash a . i \Downarrow S_{5} ; p
\end{aligned}
$$

Selections into imap-closures happen at indices that are of the same length as the concatenation of the imap frame and cell shapes. This means that the index the imap-closure is being selected from has to be split into frame and cell sub-indices: $\vec{\imath}$ and $\vec{\jmath}$ correspondingly. Given that $\bar{g}_{k}$ contains $\vec{l}$, we evaluate $e_{k}$ with $x_{k}$ being bound to $\vec{\imath}$. As this value may be non-scalar, we evaluate a selection into
it at $\vec{\jmath}$. Finally, the evaluated generator expression is saved within the imap closure. This step is performed by the helper function UpdateIMap, which splits the $k$-th partition into a single-element partition containing $\vec{\imath}$ with the computed value $p_{\vec{i}}$, and further partitions covering the remaining indices of $\bar{g}_{k}$ with the expression $e_{k}$. For more details see [58, Sec. 2.1.1].
With this, we can define and use infinite arrays in an overall strict setting. Let us consider the definitions of the infinite array of natural numbers in $\lambda_{\alpha}^{\infty}$ on the left and Haskell-like definition on the right:

```
nats \equiv imap [\infty] { _(iv): iv.[0] nats = 0: map (+1) nats
```

Both versions define an object that delivers the value $n$ when being selected at any index $n$. Both definitions provide a data structure whose computation unfolds in a lazy fashion. The main difference is that the Haskell definition enforces a left-to-right unfolding of the list. Whenever an element $n$ is selected, the entire spine of the list, up to the $n$-th element, has to be in place. In the $\lambda_{\alpha}^{\infty}$ case, any element can be computed directly. The actual access time as well as the storage demand depend on how the Imap-Lazy-rule is being implemented. In particular, it depends on how the imap-closure is being updated by an implementation of the UpdateIMap operation.

The above comparison demonstrates the fundamental difference between a data-parallel programming style and a list-based, inherently recursive programming style. Even if the former is mimicked by the latter using list comprehensions, e.g. nats $=[i \mid i<-[0 .]$.$] , the idiom [0..] boils$ down to a recursive construction of the spine of the list.

Having observed this fundamental difference, one may wonder if these kinds of Haskell-like recursive definitions are possible in $\lambda_{\alpha}^{\infty}$ at all?

### 2.5 Recursive Definitions

It turns out that the lazy imap, together with the letrec construct, allows for recursive definitions of arrays. A recursive definition of the natural numbers, including 0 , can be defined in $\lambda_{\alpha}^{\infty}$ by:
letrec nats $=\operatorname{imap}[\infty]\{[0]<=\operatorname{iv}<[1]: 0$,
$[1]<=$ iv $<[\infty]:$ nats. $(i v \dot{-}[1])+1$ in nats
The interesting question here is whether the semantics defined thus far ensures that all elements of the array nats are actually being inserted into one and the same imap-closure. For this to happen, we need the environment of the imap-closure to map nats to itself, and we need the selection within the body of the imap to modify the closure from which it is selecting. While the latter is given through the Sel-Lazy-Imap-rule, the former is achieved through the rule for letrec-constructs. For $\lambda_{\alpha}$, we have:

Letrec

$$
\begin{gathered}
S_{1}=S, p \mapsto \perp \\
\rho_{1}=\rho, x \mapsto p \quad S_{1} ; \rho_{1} \vdash e_{1} \Downarrow S_{2} ; p_{2} \quad S_{3}=S_{2}\left[p_{2} / p\right] \quad S_{3} ; \rho, x \mapsto p_{2} \vdash e_{2} \Downarrow S_{4} ; p_{r} \\
\hline S ; \rho \vdash \text { letrec } x=e_{1} \text { in } e_{2} \Downarrow S_{4} ; p_{r}
\end{gathered}
$$

where $S\left[p_{2} / p\right]$ denotes substitution of the $x \mapsto p$ bindings inside of the enclosed environments with $x \mapsto p_{2}$, where $x$ is any legal variable name. This substitution is key for creating the circular reference in the imap-closure from the example above.

In conclusion, the above recursive specification denotes an array with the same elements as the data-parallel specification from the previous section. In contrast to data-parallel version, this specification behaves much more like the recursive, Haskell-like version; the computation of individual elements can no longer happen directly. Since there is an encoded dependency between an element and its predecessor, the first access to an element at index $n$, in this variant, will trigger
the computation of all elements from 0 up to $n$. The implementation of the UpdateIMap operation on imap-closures determines how these numbers are stored in memory and, consequently, how efficiently they can be accessed.

The availability of direct indexes makes it possible to encode an arbitrary order for the recursion. Consider the following example:

```
letrec a = imap [10] { [9] <= iv < [10]: 9,
    [0] <= iv < [9]: a.(iv+[1])-1 in a
```

Selection of the 9th element can be evaluated in one step. In case of lists, the selection request always starts at the beginning of the list. Hence, to obtain the same performance, some optimisation of the list case is required.

### 2.6 List Primitives in the Array Setting

We have enabled two features that are inherent with lists, but that are usually not supported in an array setting: recursively defined data-structures and infinite arrays. All that is required to achieve this is a recursion-aware, lazy semantics of the imap-construct and the inclusion of an explicit notion of infinity. With these extensions, the key primitives for lists, head, tail, and cons can be defined as

```
head \equiv \lambdaa.a.[0]
tail \equiv\lambdaa.imap |a|\doteq[1] { _(iv): a.([1] +iv)
cons \equiv\lambdaa.\lambdab.imap [1]+|b|}{[0]<= iv < [1]: a,
    [1] <= iv < [1] +|b|: b.(iv -[1])
```

More complex list-like functions can be defined on top of these. An example is concatenation:

```
letrec (++) = \lambdaa.\lambdab.if |a|.[0] = 0 then b
    else cons (head a) ((tail a) ++ b) in (++)
```

In case $a$ is infinite, however, the above definition of concatenation is unsatisfying. The strict nature of $\lambda_{\alpha}$ will force tail a forever as $|a| .[0]=0$ never yields true. The way to avoid this is to shift the case distinction into the lazy imap construct:

$$
(++) \equiv \lambda \mathrm{a} \cdot \lambda \mathrm{~b} \cdot \operatorname{imap}|\mathrm{a}| \dot{+}|\mathrm{b}|\left\{\begin{array}{r}
{[0]<=\mathrm{iv}<|\mathrm{a}|: \mathrm{a} \cdot \mathrm{iv},} \\
|\mathrm{a}|<=\mathrm{iv}<|\mathrm{a}| \dot{+}|\mathrm{b}|: \mathrm{b} \cdot(\mathrm{iv} \dot{-}|\mathrm{a}|)
\end{array}\right.
$$

As we have seen earlier, $\lambda_{\alpha}$ enables the typical constructions of recursive definitions of infinite vectors well-known from the realm of lists such as list of ones, natural numbers or fibonacci sequence.

Having a unified interface for arrays and lists enables programmers to switch the algorithmic definitions of individual arrays from recursive to data-parallel styles without modifying any of the code that operates on them.

However, such a unification comes at a price: we have to support a lazy version of the imapconstruct. As a consequence, we conceptually lose the advantage of $O(1)$ access. Despite $\lambda_{\alpha}$ offering many opportunities for compiler optimisations like pre-allocating arrays and potentially enforcing strictness on finite, non-recursive imaps, one may wonder at this point how much $\lambda_{\alpha}$ differs from a lazy array interface in a lazy, list-based language such as Haskell?

## 3 TRANSFINITE ARRAYS

We now investigate to what extent $\lambda_{\alpha}^{\infty}$ adheres to the key properties of array programming - array algebras and array equalities.

### 3.1 Algebraic Properties

Array-based operations offer a number of beneficial algebraic properties. Typically, these properties manifest themselves as universally valid equalities which, once established, improve our thinking about algorithms and their implementations, and give rise to high-level program transformations. We define equality between two non-scalar arrays $a$ and $b$ as

$$
a==b \Longleftrightarrow|a|=|b| \wedge \forall i v<|a|: a . i v=b . i v
$$

that is, we demand equality of the shapes and equality of all elements. The demand for equality of shapes recursively implies equality in dimensionality and the extensional character of this definition through the use of array selections ensures that we can reason about equality on infinite arrays as well.

Arrays give rise to many algebras such as Theory of Arrays [46], Mathematics of Arrays [48], and Array Algebras [21]. Most of the developed algebras differ only slightly, and the set of equalities that are ultimately valid depends on some fundamental choices, such as the ones we made in the beginning of the previous section. At the core of these equalities is the ability to change the shape of arrays in a systematic way without losing any of their data.

An equality from [19] that plays a key role in consistent shape manipulations is:

$$
\begin{equation*}
\text { reshape }|a|(\text { flatten } a)==a \tag{1}
\end{equation*}
$$

where flatten maps an array recursively into a vector by concatenating its sub-arrays in a rowmajor fashion and reshape performs the dual operation of bringing a row-major linearisation back into multi-dimensional form. These operations can be defined in $\lambda_{\alpha}^{\infty}$ as

```
flatten \equiv \lambdaa.imap [count a] {_(iv): a.(o2i iv.[0] |a|)
reshape \equiv \lambdashp.\lambdaa.imap shp {__(iv): (flatten a).[i2o iv shp]
```

where count returns the product of all shape components and $o 2 i$ and $i 2 o$ translate offsets into indices and vice versa, respectively. These operations effectively implement conversions from mixedradix systems into natural numbers using multiplications and additions and back using division and remainder operations.

The above equality states that any array $a$ can be brought into flattened form and, subsequently be brought back to its original shape. For arrays of finite shape $s$, this follows directly from the fact that o2i (i2o ivs) $s=i v$ for all legitimate index vectors $i v$ into the shape $s$.

If we want Eq. 1 to hold for all arrays in $\lambda_{\alpha}^{\infty}$, we need to show that the above equality also holds for arrays with infinite axes. Consider an array of shape $s=[2, \infty]$. For any legal index vector $[1, n]$ into the shape $s$, we obtain:

$$
\begin{aligned}
o 2 i(i 2 o[1, n][2, \infty])[2, \infty]) & =o 2 i(\infty \cdot 1+n)[2, \infty] \\
& =o 2 i \infty[2, \infty] \\
& =[\infty / \infty, \infty \% \infty]
\end{aligned}
$$

which is not defined. We can also observe that all indices $[1, n]$ are effectively mapped into the same offset: $\infty$ which is not a legitimate index into any array in $\lambda_{\alpha}^{\infty}$. This reflects the intuition that the concatenation of two infinite vectors effectively looses access to the second vector.

The inability to concatenate infinite arrays also makes the following equality fail:

$$
\begin{equation*}
\operatorname{drop}|a|(a++b)==b \tag{2}
\end{equation*}
$$

where $a$ and $b$ are vectors and drops $x$ removes first $s$ elements from the left. The reason is exactly the same: given that $|a|=[\infty]$ and $b$ is of finite shape $[n]$, the shape of the concatenation is $[\infty+n]=[\infty]$, and drop of $|a|$ results in an empty vector.

Clearly, $\lambda_{\alpha}^{\infty}$ as presented so far is not strong enough to maintain universal equalities such as Eq. 1 or 2. Instead, we have to find a way that enables us to represent sequences of infinite sequences that can be distinguished from each other.

### 3.2 Ordinals

When numbers are treated in terms of cardinality, they describe the number of elements in a set. Addition of two cardinal numbers $a$ and $b$ is defined as a size of a union of sets of $a$ and $b$ elements. This notion also makes it possible to operate with infinite numbers, where the number of elements in an infinite set is defined via bijections. As a result, differently constructed infinite sets may end up having the same number of elements. For example, if there exists a bijection from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$, the cardinality of both sets is the same.

When studying arrays, treating their shapes and indices using cardinal numbers is an oversimplification, because arrays have richer structure. Arrays are collections of ordered elements, where the order is established by the indices. Ordinal numbers, as introduced by G. Cantor in 1883, serve exactly this purpose - to "label" positions of objects within an ordered collection. When collections are finite, cardinals and ordinals can be used interchangeably, as we can always count the labels. Infinite collections are quite different in that regard: despite being of the same size, there can be many non-isomorphic well-orderings of an infinite collection. For example, consider two infinite arrays of shapes $[\infty, 2]$ and $[2, \infty]$. Both of these have infinitely many elements, but they differ in their structure. From a row major perspective, the former is an infinite sequence of pairs, whereas the latter are two infinite sequences of scalars. Ordinals give a formal way of describing such different well-orderings.

First let us try to develop an intuition for the concept of ordinal numbers and then we give a formal definition. Consider an ordered sequence of natural numbers: $0<1<2<\cdots$. These are the first ordinals. Then, we introduce a number called $\omega$ that represents the limit of the above sequence: $0<1<2<\cdots<\omega$. Further, we can construct numbers beyond $\omega$ by putting a "copy" of natural numbers "beyond" $\omega$ :

$$
0<1<2<\cdots \omega<\omega+1<\omega+2<\cdots<\omega+\omega
$$

For the time being, we treat operations such as $\omega+n$ symbolically. The number $\omega+\omega$ which can be also denoted as $\omega \cdot 2$ is the second limit ordinal that limits any number of the form $\omega+n, n \in \mathbb{N}$. Such a procedure of constructing limit ordinals out of already constructed smaller ordinals can be applied recursively. Consider a sequence of $\omega \cdot n$ numbers and its limit:

$$
0<\omega<\omega \cdot 2<\omega \cdot 3<\cdots<\left(\omega \cdot \omega=\omega^{2}\right)
$$

and we can carry on further to $\omega^{n}, \omega^{\omega}$, etc. Note though that in the interval from $\omega^{2}$ to $\omega^{3}$ we have infinitely many limit ordinals of the form:

$$
\omega^{2}<\omega^{2}+\omega<\omega^{2}+\omega \cdot 2<\cdots<\omega^{3}
$$

and between any two of these we have a "copy" of the natural numbers:

$$
\omega^{2}+\omega<\omega^{2}+\omega+1<\cdots<\omega^{2}+\omega \cdot 2
$$

3.2.1 Formal definitions. A totally ordered set $\langle A,<\rangle$ is said to be well ordered (or have a wellfounded order) if and only if every nonempty subset of $A$ has a least element [16]. Given a wellordered set $\langle X,<\rangle$ and $a \in X, X_{a} \stackrel{\text { def }}{=}\{x \in X \mid x<a\}$. An ordinal is a well-ordered set $\langle X,<\rangle$, such that: $\forall a \in X: a=X_{a}$. As a consequence, if $\langle X,<\rangle$ is an ordinal then $<$ is equivalent to $\in$. Given a well-ordered set $A=\langle X,<\rangle$ we define an ordinal that this set is isomorphic to as $\operatorname{Ord}(A,<)$. Given
an ordinal $\alpha$, its successor is defined to be $\alpha \cup\{\alpha\}$. The minimal ordinal is $\emptyset$ which is denoted with 0 . The next few ordinals are:

$$
\begin{aligned}
1=\{0\} & =\{\emptyset\} \\
2=\{0,1\} & =\{\emptyset,\{\emptyset\}\} \\
3 & =\{0,1,2\}=\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}
\end{aligned}
$$

A limit ordinal is an ordinal that is greater than zero that is not a successor. The set of natural numbers $\{0,1,2,3, \ldots\}$ is the smallest limit ordinal that is denoted $\omega$. We use islim $x$ to denote that $x$ is a limit ordinal.

### 3.2.2 Arithmetic on Ordinals.

Addition. Ordinal addition is defined as $\alpha+\beta=\operatorname{Ord}\left(A,{ }_{A}\right)$, where $A=\{0\} \times \alpha \cup\{1\} \times \beta$ and $<_{A}$ is the lexicographic ordering on $A$. Ordinal addition is associative but not commutative. As an example consider expressions $2+\omega$ and $\omega+2$. The former can be seen as follows: $0<$ $1<0^{\prime}<1^{\prime}<\cdots$, which after relabeling is isomorphic to $\omega$. However, the latter can be seen as: $0<1<\cdots<0^{\prime}<1^{\prime}$, which has the largest element $1^{\prime}$, whereas $\omega$ does not. Therefore $2+\omega=\omega<\omega+2$. We have used $0^{\prime}, 1^{\prime}$ to indicate the right hand side argument of the addition.

Subtraction. Ordinal subtraction can be defined in two ways, as partial inverse of the addition on the left and on the right. For left subtraction, which will be used by default throughout this paper unless otherwise specified, $\alpha-\beta$ is defined when $\beta \leq \alpha$, as: $\exists \xi: \beta+\xi=\alpha$. As ordinal addition is left-cancelative $(\alpha+\beta=\alpha+\gamma \Longrightarrow \beta=\gamma)$, left subtraction always exists and it is unique.

Right subtraction is a bit harder to define as:

- it is not unique: $1+\omega=2+\omega$ but $1 \neq 2$; therefore $\omega-_{R} \omega$ can be any number that is less than $\omega:\{0,1,2, \ldots\}$.
- even if $\beta<\alpha$, the difference $\alpha-\beta$ might not exist. For example: $42<\omega$; however, $\omega-{ }_{R} 42$ does not exist as $\nexists \xi: \xi+42=\omega$.
Despite those difficulties, right subtraction can be useful at times and can be defined for $\alpha{ }_{-} \beta$ :

$$
\min \{\xi: \xi+\beta=\alpha\}
$$

Multiplication. Ordinal multiplication $\alpha \cdot \beta=\operatorname{Or} d\left(A,<_{A}\right)$ where $A=\alpha \times \beta$ and $<_{A}$ is the lexicographic ordering on $A$. Multiplication is associative and left-distributive to addition:

$$
\alpha \cdot(\beta+\gamma)=(\alpha \cdot \beta)+(\alpha \cdot \gamma)
$$

However, multiplication is not commutative and is not distributive on the right: $2 \cdot \omega=\omega<\omega \cdot 2$ and $(\omega+1) \cdot \omega=\omega \cdot \omega<\omega \cdot \omega+\omega$.

Exponentiation. Exponentiation can be defined using transfinite recursion: $\alpha^{0}=1, \alpha^{\beta+1}=\alpha^{\beta} \cdot \alpha$ and for limit ordinals $\lambda: \alpha^{\lambda}=\underset{0<\xi<\lambda}{\bigcup} \alpha^{\xi}$.
$\epsilon$-ordinals. Using ordinal operations we can construct the following hierarchy of ordinals: $0,1, \ldots, \omega, \omega+$ $1, \ldots, \omega \cdot 2, \omega \cdot 2+1, \ldots, \omega^{2}, \ldots, \omega^{3}, \ldots \omega^{\omega}, \ldots$. The smallest ordinal for which $\alpha=\omega^{\alpha}$ is called $\epsilon_{0}$. It can also be seen as a limit of the following $\omega^{\omega}, \omega^{\omega^{\omega}}, \ldots, \omega^{\omega \cdots}$.
3.2.3 Cantor Normal Form. For every ordinal $\alpha<\epsilon_{0}$ there are unique $n, p<\omega, \alpha_{1}>\alpha_{2}>\cdots>$ $\alpha_{n}$ and $x_{1}, \ldots, x_{n} \in \omega \backslash\{0\}$ such that $\alpha>\alpha_{1}$ and $\alpha=\omega^{\alpha_{1}} \cdot x_{1}+\cdots+\omega^{\alpha_{n}} \cdot x_{n}+p$. Cantor Normal Form makes provides a standardized way of writing ordinals. It uniquely represents each ordinal
as a finite sum of ordinal powers, and can be seen as an $\omega$ based polynomial. This can be used as a basis for an efficient implementation of ordinals and their operations.

## $3.3 \lambda_{\omega}$ : Adding Ordinals to $\lambda_{\alpha}$

The key contribution of this paper is the introduction of $\lambda_{\omega}$, a variant of $\lambda_{\alpha}$, which use ordinals as shapes and indices of arrays and which reestablishes global equalities in the context of infinite arrays.

Before revisiting the equalities, we look at the changes to $\lambda_{\alpha}$ that are required to support transfinite arrays. Syntactically, to introduce ordinals in the language, we make two minor additions to $\lambda_{\alpha}$. Firstly, we add ordinals ${ }^{4}$ as scalar constants. Secondly, we add a built-in operation, islim, which takes one argument and returns true if the argument is a limit ordinal and false otherwise. For example: islim $\omega$ reduces to true and islim $(\omega+21)$ reduces to false.


Fig. 3. The syntax of $\lambda_{\omega}$.
Semantically, it turns out that all core rules can be kept unmodified apart from the aspect that all helper functions, arithmetic, and relational operations now need to be able to deal with ordinals instead of natural numbers. In particular, the semantic for lazy imaps as developed for $\lambda_{\alpha}^{\infty}$ can be used unaltered, provided that all helper functions involved such as for splitting generators etc. are expanded to cope with ordinals.

### 3.4 Array Equalities Revisited

With the support of Ordinals in $\lambda_{\omega}$, we can now revisit our equalities Eq. 1 and 2. Let us first look at the counter example for Eq. 1: from Section 3.1: With an array shape $s=[2, \omega]$ and a legal index vector into $s[1, n]$, we now obtain:

$$
\begin{aligned}
o 2 i(i 2 o[1, n][2, \omega])[2, \omega]) & =o 2 i(\omega+n)[2, \omega] \\
& =[(\omega+n) / \omega,(\omega+n) \% \omega] \\
& =[1, n]
\end{aligned}
$$

The crucial difference to the situation from $\lambda_{\alpha}^{\infty}$ in Section 3.1 here is the ability to divide $(\omega+n)$ by $\omega$ and to obtain a remainder, denoted by $\%$, of that division as well. By means of induction over the length of the shape and index vectors this equality can be proven to hold for arbitrary shapes in $\lambda_{\omega}$, and, based on this proof, Eq. 1 can be shown as well.

In the same way as the arithmetic on ordinals is key to the proof of Eq. 1, it also enables the proof of Eq. 2 for arbitrary ordinal-shaped vectors ${ }^{5} a$ and $b$, with the definition of ++ from the previous section and drop being defined as:

```
drop \equiv \lambdas.\lambdaa. imap |a|̇s { [0] <= iv < |a| - s: a.(s+iv)
```

[^2]After inlining ++ and drop, the left hand side of Eq. 2 can be rewritten as:

```
letrec \(a b=\operatorname{imap}|a|+|b|\{[0]<=j v<|a|: a \cdot j v\),
    \(|a|<=j v<|a|+|b|: b .(j v \dot{-}|a|)\) in
\(\operatorname{imap}|a b|-|a|\{[0]<=i v<|a b| \dot{-}|a|: a b .(|a|+i v)\)
```

Consider the shape of the goal expression of the letrec. According to the semantics of the shape of an imap, we get: $|a b| \dot{-}|a|$. The shape of $a b$ is $|a| \dot{+}|b|$. According to ordinal arithmetic: $(|a| \dot{+}|b|) \dot{-}|a|$ is $|b|$. Therefore the shapes of right-hand and left-hand sides of the goal expressions are the same.

Let us rewrite the last imap as:
$\operatorname{imap}|b|\{[0]<=i v<|b|: a b .(|a|+i v)$
Consider now selections into $a b$. All the selections into $a b$ will happen at indices that are greater than $a$. This is because all the legal $i v$ in the imap are from the range [0] to $|b|$.

According to the semantics of selections into imaps, $a b .(|a| \dot{+} i v)$ will select from the second partition of the imap that defines $a b$, and will evaluate to: $b \cdot((|a| \dot{+} i v) \dot{-}|a|)$. According to ordinal arithmetic, $(|a| \dot{+} i v) \dot{-}|a|$ is identical to $i v$, therefore we can rewrite the previous imap as:

```
imap |b| {[0]<= iv < |b|: b.iv
```

As it can be seen, this is an identity imap, which is extensionally equivalent to $b$.

## 4 EXAMPLES

Transfinite tail. As explained in Section 3.3, the shift from natural numbers to ordinals as indices in $\lambda_{\omega}$ implies corresponding extensions of the built-in arithmetic operations. As these operations lose key properties, such as commutativity, once arguments exceed the range of natural numbers, we need to ensure that function definitions for finite arrays extend correctly to the transfinite case.

As an example, consider the definition of tail from the previous section:

```
tail \equiv \lambdaa.imap }|\textrm{a}|\subset[1] {_(iv): a.([1] +ivv)
```

For the case of finite vectors, we can see that a vector shortened by one element is returned, where the first element is missing and all elements have been shifted to the left by one element.

Let us assume we apply tail to an array $a$ with $|a|=[\omega]$. The arithmetic on ordinals gives us a return shape of $[\omega] \dot{-}[1]=[\omega]$. That is, the tail of an infinite array is the same size as the array itself, which matches our common intuition when applying tail to infinite lists. The elements of that infinite list are those of $a$, shifted by one element to the right, which, again, matches our expected interpretation for lists.
Now, assume we have $|a|=[\omega+42]$, which means that (tail a). $[\omega]$ should be a valid expression. For the result shape of tail $a$, we obtain $[\omega+42] \dot{-}[1]=[\omega+42]$. A selection (tail $a)$. $[\omega]$ evaluates to $a .([1]+[\omega])=a .[\omega]$. This means that the above definition of the tail shifts all the elements at indices smaller than $[\omega]$ one left, and leaves all the other unmodified. While this may seem counterintuitive at first, it actually only means that tail can be applied infinitely often but will never be able to reach "beyond" the first limit.

Finally, observe that the body of the imap-construct in the definition of tail uses [1] $\dot{+i v}$ is an index expression, not $i v+[1]$. In the latter case, the tail function would behave differently beyond $[\omega]$ : it would attempt to shift elements to the left. However, this would make the overall definition faulty. Consider again the case when $|a|=[\omega+42]$ : the shape of the result would be $|a|$, which would mean that it would be possible to index at position [ $\omega+41$, triggering evaluation of $a$. ( $[\omega+$ 41] $+[1]$ ) and consequently, producing an index error, or out-of-bounds access into $a$.

Zip. Let us now define zip of two vectors that produces a vector of tuples. Consider a Haskell definition of zip function first:

```
zip (a:as) (b:bs) = (a,b): zip as bs
zip _ _ = []
```

The result is computed lazily, and the length of the resulting list is a minimum of the lengths of the arguments. Like concatenation, a literal translation into $\lambda_{\omega}$ is possible, but it has the same drawbacks, i.e. it is restricted to arrays whose shape has no components bigger than $\omega$.

A better version of zip that can be applied to arbitrary transfinite arrays looks as follows:

```
zip \equiv \lambdaa.\lambdab.imap (min |a| |b|)|[2] {_(iv): [a.iv, b.iv]
```

Here, we use a constant array in the body of the imap. This forces evaluation of both arguments, even if only one of them is selected. This can be changed by replacing the constant array with an imap:

```
zip \equiv \lambdaa.\lambdab.imap (min |a| |b|)|[2] {_(iv): imap [2] { [0] <= jv < [1] a.iv,
    [1] <= jv < [2] b.iv
```

which can be fused in a single imap as follows:

```
zip }\equiv\lambda\textrm{a}\cdot\lambda\textrm{b}.l(\textrm{letrec}\textrm{s}=(\operatorname{min}|\textrm{a}||\textrm{b}|)\cdot[0] i
    imap [s,2] {[0,0]<= iv < [s, 1]: a.[iv.[0]],
    [0,1] <= iv < [s, 2]: b.[iv.[0]]
```

Data Layout and Transpose. A typical transformations in stream programming is changing the granularity of a stream and joining multiple streams. In $\lambda_{\omega}$, these transformations can be expressed by manipulating the shape of an infinite array. Consider changing the granularity of a stream $a$ of shape $[\omega]$ into a stream of pairs:

```
imap (|a|j[2])|[2] { _(iv): [a.[2*iv.[0]], a.[2*iv.[0]+1]]
```

or we can express the same code in a more generic fashion:

```
(\lambdan.reshape ((|a|/[n])++[n]) a) 2
```

This code can operate on the streams of transfinite length, as well. If we envision compiling such a program into machine code, the infinite dimension of an array can be seen as a time-loop, and the operations at the inner dimension seen as a stream-transforming function. Such granularity changes are often essential for making good use of (parallel) hardware resources, e.g. FPGAs.

Transposing a stream makes it possible to introduce synchronisation. Consider transforming a stream $a$ of shape $[2, \omega]$ into a stream of pairs (shape $[\omega, 2]$ ):

```
imap [\omega]|[2] {_(iv): [a.[iv.[0],0], a.[iv.[0],1]]
```

Conceptually, an array of shape $[2, \omega]$ represents two infinite streams that reside in the same data structure. An operation on such a data structure can progress independently on each stream, unless some dependencies on the outer index are introduced. A transpose, as above, makes it possible to introduce such a dependency, ensuring that the operations on both streams are synchronized. A key to achieving this is the ability to re-enumerate infinite structures, and ordinal-based infinite arrays make this possible.

Ackermann function. The true power of multidimensional infinite arrays manifests itself in definitions of non-primitive-recursive sequences as data. Consider the Ackermann function, defined as a multi-dimensional stream:

```
letrec \(a=\operatorname{imap}[\omega, \omega]\left\{\_(i v):\right.\) letrec \(m=i v .[0]\) in
    letrec \(n=i v .[1]\) in
    if \(m=0\) then \(n+1\)
    else if \(m>0\) and \(n=0\) then a. \([m-1,1]\)
    else a. \([\mathrm{m}-1, \mathrm{a} \cdot[\mathrm{m}, \mathrm{n}-1]]\) in a
```

Such a treatment of multi-dimensional infinite structures enables simple transliteration of recursive relations as data. Achieving similar recursive definitions when using cons-lists is possible, but they have a subtle difference. Consider a Haskell definition of the Ackermann function in data:

```
\(\mathrm{a}=\) [ \([\) if \(\mathrm{m}==0\) then \(\mathrm{n}+1\)
        else if \(\mathrm{m}>0\) then a !! (m-1) !! 1
        else a !! (m-1) !! (a !! m !! (n-1))
    | \(\mathrm{n}<-\) [0..]]
    | \(\mathrm{m}<-\) [0..]]
```

We use two [0..] generators for explicit indexing, even though at runtime, all necessary elements of the list will be present. The lack of explicit indexes forces one to use extra objects to encode the correct dependencies, essentially implementing imap in Haskell. Conceptually, these generators constitute two further locally recursive data structures. Whether they can be always can be optimised away is not clear. Avoiding these structures in an algorithmic specification can be a major challenge.

Game of Life. As a final example, consider Conway's Game of Life. First we introduce a few generic helper functions:

```
(or) \equiv\lambdaa.\lambdab.if a then a else b
(and) \equiv \lambdaa.\lambdab.if a then b else a
any \equiv\lambdaa.reduce or false a
gen \equiv \s.\lambdav.imap s {_(iv): v
\ \equiv\lambdav.\lambdaa.imap |a| {_(iv): if any (iv\dot{+}>>=|a|) then 0 else a.(iv\dot{+v})
\searrow \equiv\lambdav.\lambdaa.imap |a| {_(iv): if any (iv < v) then 0 else a.(iv\dot{-v})
```

or and and encode logical conjunction and disjunction, respectively. any folds an array of boolean expressions with the disjunction, and gen defines an array of shape $s$ whose values are all identical to $v$. More interesting are the functions $\nwarrow$ and $\searrow$. Given a vector $v$ and an array $a$, they shift all elements of $a$ towards decreasing indices or increasing indices by $v$ elements, respectively. Missing elements are treated as the value 0 .

Now, we define a single step of the 2-dimensional Game of Life in APL style ${ }^{6}$ : two-dimensional array $a$ by:

```
gol_step \(\equiv \lambda \mathrm{a}\).
    letrec \(F=[\nwarrow[1,1], \nwarrow[1,0], \nwarrow[0,1], \lambda x . \nwarrow[1,0](\searrow[0,1] x)\),
    in \(\searrow[0,1], \searrow[1,0], \searrow[1,1], \lambda x . \searrow[1,0](\nwarrow[0,1] x)]\)
        letrec
        \(\mathrm{c}=(\) reduce \((\lambda \mathrm{f} . \lambda \mathrm{g} . \lambda \mathrm{x} . \mathrm{f} \mathrm{x} \dot{+} \mathrm{g} \mathrm{x})(\lambda \mathrm{x} . \mathrm{gen}|\mathrm{a}| 0) \mathrm{F}) \mathrm{a}\)
    in
        \(\operatorname{imap}|\mathrm{a}|\left\{\left\{_{-}(\mathrm{iv}):\right.\right.\) if \((\mathrm{c} \cdot \mathrm{iv}=2\) and \(\mathrm{a} \cdot \mathrm{iv}=1)\) or \((\mathrm{c} \cdot \mathrm{iv}=3)\)
                        then 1
                        else 0
```

We assume an encoding of a live cell in $a$ to be 1 , and a dead cell to be 0 . The array $F$ contains partial applications of the two shift functions to two-element vectors so that shifts into all possible directions are present. The actual counting of live cells is performed by a function which folds $F$ with the function $\lambda f . \lambda g . \lambda x . f x+g x$. This produces $c$, an array of the same shape as $a$, holding the

[^3]numbers of live cells surrounding each position. Defining the shift operations $\nwarrow$ and $\searrow$ to insert 0 ensures that all cells beyond the shape of $a$ are assumed to be dead.

The definition of the result array is, therefore, a straightforward imap, implementing the rules of birth, survival and death of the Game of Life.

The most interesting aspect of this algorithm is the fact that there is no restriction on the shape of $a$. In our transfinite setting, we can provide an array of shape $[\omega, \omega]$. With no changes to source code, we can deal with an infinitely large plane. An infinite $a$ requires a lazy implementation as demanded by our semantics of $\lambda_{\omega}$, but a finite case offers a strict implementation as a possible alternative.

## 5 TRANSFINITE ARRAYS VS. STREAMS

Streams have attracted a lot of attention due to the many algebraic properties they expose. [29] provides a nice collection of examples, many of which are based on the observation that streams form an applicative functor. Transfinite arrays are applicative functors as well, not only for arrays of shape $[\omega]$, but also for any given shape $s h p$. With definitions:

```
pure \(\equiv \lambda x\).imap \(\operatorname{shp}\left\{_{-}(i v): x\right.\)
\((\diamond) \equiv \lambda a \cdot \lambda b . \operatorname{imap} \operatorname{shp}\left\{\_(i v): a \cdot i v \quad b . i v\right.\)
```

we obtain for arbitrary arrays $u, v, w$, and $x$ of shape $\operatorname{shp}$ :

```
\((\) pure \(\lambda x \cdot x) \diamond u==u \quad(\) pure \((\lambda f . \lambda g \cdot \lambda x \cdot f(g x))) \diamond u \diamond v \diamond w==u \diamond(v \diamond w)\)
\((\) pure \(f) \diamond(\) pure \(x)==\operatorname{pure}(f x) \quad u \diamond(\) pure \(x)==(\) pure \((\lambda f . f x)) \diamond u\)
```

This shows that arbitrarily shaped arrays of finite size have this property, as also shown by [20], and that these properties can be expanded into ordinal-shaped arrays. Classical streams are a special instance of these, i.e. arrays of shape [ $\omega$ ].

For stream operations that insert or delete elements, it is less obvious whether these can be easily extended into ordinal-shaped arrays other than shape [ $\omega$ ]. As an example, let us consider the function filter, which takes a predicate $p$ and a vector $v$ and returns a vector that contains only those elements $x$ of $v$ that satisfy $(p x)$. A direct definition of filter can be given as:

```
filter \equiv \p. \lambdav. if (p v.[0]) then v.[0] ++ filter p (tail v)
    else filter p (tail v)
```

This definition, in principle, is applicable to arrays of any ordinal shape, but the use of tail in the recursive calls inhibits application beyond $\omega$. Furthermore, the strict semantics of $\lambda_{\omega}$ inhibits a terminating application to any infinite array, including arrays of shape $[\omega]$. For the same reason, a definition of filter through the built-in reduce is restricted to finite arrays.

To achieve possible termination of the above definition of filter for transfinite arrays, we would need to change to a lazy regime for all function applications in $\lambda_{\omega}$ and we would need to change the semantics of imap into a variant where the shape computation can be delayed as well. Even if that would be done, we would still end up with an unsatisfying solution. The filtering effect would always be restricted to the elements before the first limit ordinal $\omega$. This limitation breaks several fundamental properties, like those defined in [10], that hold in the finite and stream cases. As an example, consider distributivity of filter over concatenation:

$$
\begin{equation*}
\text { filter } p(a++b)==(\text { filter } p a)++(\text { filter } p b) \tag{3}
\end{equation*}
$$

This property holds for finite arrays, but fails with the above definition of filter in case $a$ is infinite.
To regain this property for transfinite arrays, we need to apply filter to all elements of the argument array, not only those before the first limit ordinal $\omega$. When doing this in the context of $\lambda_{\omega}$, the necessity to have a strict shape for every object forces us to "guess" the shape of the filtered
result in advance. The way we "guess" has an impact on the filter-based equalities that will hold universally.

In this paper we propose a scheme that respects the above equality. For finite arrays filter works as usual, and for the infinite ones, we postulate that the result of filtering will be of an infiniteshape:

$$
\forall p \forall a:|a| \geq \omega \Longrightarrow \mid \text { filter } p a \mid \geq \omega
$$

This is further applied to all infinite sequences contained within the given shape as follows:

$$
\forall i<|a|:(\exists i s \lim \alpha: i<\alpha \leq|a|) \Longrightarrow(\exists k \in \mathbb{N}: p(a .(i+k))=\text { true })
$$

We assume that each infinite sequence contains infinitely many elements for which the predicate holds. Consequently, any limit ordinal component of the shape of the argument is carried over to the result shape and only any potential finite rest undergoes potential shortening. Consider a filter operation, applied to a vector of shape $[\omega \cdot 2]$. Following the above rationale, the shape of the result will be $[\omega \cdot 2]$ as well. This means that the result of applying filter to such an expression should allow indexing from $\{0,1, \ldots\}$ as well as from $\{\omega, \omega+1, \ldots\}$ delivering meaningful results.

This decision can lead to non-termination when there are only finitely many elements in the filtered result. For example:
filter $(\lambda x . x>0)\left(\operatorname{imap}[\omega+2] \quad\left\{\_(i v): 0\right)\right.$
reduces to an array of shape [ $\omega$ ], which effectively is empty. Any selection into it will lead to a non-terminating recursion.

The overall scheme may be counter-intuitive, but it states that for every index position of the output, the computation of the corresponding value is well-defined.

Assuming the aforementioned behaviour of filter, Eq. 3 holds for all transfinite arrays. Another universal equation that holds for all transfinite vectors concerns the interplay of filter and map:

$$
\text { filter } p(\operatorname{map} f a)==\operatorname{map} f(\text { filter }(p \cdot f) a)
$$

The proposed approach does not only respect the above equalities, but it also behaves similarly to filtering of streams that can be found in languages such as Haskell: filter applied to an infinite stream cannot return a finite result.

In principle, the chosen filtering scheme can be defined in $\lambda_{\omega}$ by using the islim predicate within an imap. However, the resulting definition is neither concise, nor likely to be runtime efficient. Given the importance of filter, we propose an extension of $\lambda_{\omega}$. Fig. 4 shows the syntactical extension of $\lambda_{\omega}$.


Fig. 4. The syntax of $\lambda_{\omega}$ with filters.

As filter conceptually is an alternative means of constructing arrays, its semantics is similar to that of imap. In particular, it constitutes a lazy array constructor, whose elements are being evaluated upon demand created through selections. Technically, this means that we have to introduce a new value to keep filter-closures, a rule that builds such a closure from filter expression, and we need to define the selection operation that forces evaluation within the filter closure.

We introduce as new value for filter-closures:

$$
\| \text { filter } p_{f} p_{e}\left\{\begin{array}{ll}
\alpha_{1} & v_{r}^{1} v_{i}^{1} \\
\cdots & \\
\alpha_{n} & v_{r}^{n} \\
v_{i}^{n}
\end{array} \|\right.
$$

which contains the pointer to the filtering function $p_{f}$, the shape of the argument we are filtering over $\left(p_{e}\right)$ and the list of partitions that consist of a limit ordinal, and a pair of partial result and natural number: $v_{r}$ and $v_{i}$ correspondingly.

On every selection at index $[\xi+n]$, where $\xi$ is a limit ordinal or zero, and $n$ is a natural number, we find a $\xi$ partition within the filter closure or add a new one if it is not there. Every partition keeps a vector with a partial result of filtering $\left(v_{r}\right)$, and the index $\left(v_{i}\right)$ with the following property: the element in the array we are filtering over at position $\xi+\left(v_{i}-1\right)$ is the last element in the $v_{r}$, given that $v_{r}>0$. This means that if $n$ is within $v_{r}$, we return $v_{r}$. [n]. Otherwise, we extend $v_{r}$ until its length becomes $n+1$ using the following procedure: inspect the element in $p_{e}$ at the position $\xi+v_{i}$ - if the predicate function evaluates to true, append this element to $v_{r}$ and increase $v_{i}$ by one, otherwise, increase $v_{i}$ by one.

A formal description of this procedure can be found in [58, Sec. 2.1.4].

## 6 TOWARDS AN IMPLEMENTATION

We used the semantics of $\lambda_{\omega}$ as a blueprint for the implementation of an interpreter, called Heh available at https://github.com/ashinkarov/heh). The interpreter, which serves as a proof of concept, performs a literal translation of the semantic rules provided in the paper into Ocaml code. All examples provided in the paper can be found in that repository, and run, correctly, in the interpreter.

The implementation parses the program, evaluates it and prints the result. We represent the storage $S$ from our semantics by a hash table of pointer-value bindings. Environments $\rho$ are implemented as lists of variable-pointer pairs. Pointers and variables are strings and values are of an algebraic data type. In the proof-of-concept interpreter, we never actively discard pointers or variables; however we do share pointers and we update imap/filter closures in place, in the same way as it is done in the formal semantics.

We represent ordinals in Cantor Normal Form. The algorithms for implementing operations on ordinals are based on [42]. In the same paper, we also find an in-depth study of the complexities of ordinal operations: comparisons, additions and subtractions have complexities $O(n)$, where $n$ is the minimum of the lengths of both arguments; multiplications have the complexity $O(n \cdot m)$, where $m$ and $n$ are the lengths of the two argument representations.

The interpreter makes it possible to run all the examples described in this paper. Additionally, the interpreter provides means for experimentation through the incorporation of variants in the semantics of imap: two interpreter flags enable users to (i) avoid the memoization of array elements completely, and (ii) to apply the strict imap-semantics instead of the lazy one whenever arrays are of finite shape. The implementation comes with about 100 unit tests.

### 6.1 Performance considerations

Having an interpreter for $\lambda_{\omega}$ available allows experimentation with ordinal indexing and transfinite definitions. However, one of our initial aims, to enable efficient runtime execution on parallel systems, is not demonstrated by Heh. In the remainder of this section, we discuss several performance considerations that show how we envision efficient parallel executions of $\lambda_{\omega}$ to be possible.

Strictness. As mentioned in Section 2, the design of $\lambda_{\alpha}$ closely matches that of SAC which has been shown to deliver high-performance execution on a variety of parallel machine architectures [56,59]. Since $\lambda_{\omega}$ is largely an extension of $\lambda_{\alpha}$ to support infinite arrays, we expect that programs that refrain from using infinite arrays can be mapped in SAC programs and, thus, benefit from the compiler tool chain ${ }^{7}$ for getting high-performance parallel execution. A prerequisite for this is that switching from the lazy variant of the imap-construct as defined for $\lambda_{\omega}$, to the strict version of imap from $\lambda_{\alpha}$, is valid, i.e. the switch does not change the semantics of a program under consideration. A comparison of the corresponding two semantic definitions in Section 2 shows that this is legitimate, if and only if (i) the shape of the array that is constructed is finite, (ii) the array is not recursively defined, and (iii) all elements of the array are being accessed. Criterion (i) is trivial to decide. Criterion (ii), while being undecidable in general, in practice, can be approximated in most cases straightforwardly. The third criterion is more difficult to approximate by means of analyses. We identify two possible alternatives to conservative approximation:

- programmers could be allowed to explicitly annotate strictness of imap-constructs or just individual partitions of them. While this seems very effective, in principle, it comes with some drawbacks as well: if a programmer annotates too little strictness, there might be a noticeable performance penalty and any wrong annotations could lead to non-termination.
- some form of dynamic switch between strict and lazy modes of imap evaluation could be implemented, speculatively evaluating some arguments to some extent.
The ability to have mixed strict and lazy imap semantics in Heh facilitates experimentation in this regard.

Strict Recursion. Even if criterion (ii) from the previous paragraph is not given, as long as the other two criteria hold, a strict evaluation is possible but it can no longer be performed in a dataparallel style because of dependencies between the elements. Given that it is known in advance that the entire subspace of the imap needs to be evaluated, the order of traversal of the elements can dramatically impact performance of such an evaluation. If all the dependencies between the elements in a recursive imap are linear with respect to index, then such a recursive imap can be presented in the polyhedral model as a loop-nest. This would give a rise to very powerful optimisations that are well understood within polyhedral compilation frameworks. The question whether infinite specifications can be handled by the polyhedral model as efficiently as finite ones remains open, offering perspective for future work.

Data structures. The current semantics prescribe that, when evaluating selections into a lazy imap, the partition that contains the index is split into a single-element partition and the remainder. This means that, as the number of selections into the imap increases, the structure that stores partitions of the imap will have to deal with a large number of single-element arrays. Partitions can be stored in a tree, providing $O(\log n)$ look-up; however triggering a memory allocation per every scalar can be very inefficient. An alternate approach would be to allocate larger chunks, each of which would store a subregion of the index space of an imap. When doing so, we would need to establish a policy on the size of chunks and chose a mechanism on how to indicate evaluated elements in a chunk. Another possibility would be to combine the chunking with some strictness speculation, as explained in the previous paragraph. We could trigger the evaluation of the entire chunk whenever any element of the chunk is selected.

Memory management. An efficient memory management model is not obvious. In case of strict arrays, reference counting is known to be an efficient solution [14, 23]. For lazy data structures,

[^4]garbage collection is usually preferable. Most likely, the answer lies in a combination of those two techniques.

The imap construct offers an opportunity for garbage collection at the level of partitions. Consider a lazy imap of boolean values with a partition that has a constant expression:
$\operatorname{imap}[\omega]\{\ldots, 1<=\mathrm{iv}<\mathrm{u}:$ false, $\ldots$
Assume further that neighbouring partitions evaluate to false. In this case, we can merge the boundaries of partitions and instead of keeping values in memory, the partition can be treated as a generator. However, an efficient implementation of such a technique is non-trivial.

Ordinals. An efficient implementation of ordinals and their operations is also essential. Here, we could make use of the fact that $\lambda_{\omega}$ is limited to ordinals up to $\omega^{\omega}$. For further details see [58, Sec. 4]

## 7 RELATED WORK

Several works propose to extend the index domain of arrays to increase expressibility of a language. A straightforward way to do this is to stay within cardinal numbers but add a notion of $\infty$, similarly to what we have proposed in $\lambda_{\alpha}^{\infty}$. Similar approach is described in [44]; in J [35] infinity is supported as a value, but infinite arrays are not allowed. As we have seen, by doing so we lose a number of array equalities.

In [46, page 137] we read: 'A restriction of indices to the finite ordinal numbers is a needless limitation that obscures the essential process of counting and indexing.' We cannot agree more. [46] describes an axiomatic array theory that combines set theory and APL. The theory is self contained and gives rise to a number of array equalities. However, the question on how this theory can be implemented (if at all) is not discussed.

In [52] the authors propose to extend the domain of array indices with real numbers. More specifically, a real-valued function gives rise to an array in which valid indices are those that belong to the domain of that function. The authors investigate expressibility of such arrays and they identify classes of problems where this could be useful, but neither provide a full theory nor discuss any implementation-related details.

Besides the related work that stems from APL and the plethora of array languages that evolved from it, there is an even larger body of work that has its origins in lists and streams. One of the best-known fundamental works on the theory of lists using ordered pairs can be found in [43, sec. 3], where a class of S-expressions is defined. The concepts of nil and cons are introduced, as well as car and $c d r$, for accessing the constituents of cons.

The Theory of Lists [10] defines lists abstractly as linearly ordered collections of data. The empty list and operations like length of the list, concatenation, filter, map and reduce are introduced axiomatically. Lists are assumed to be finite. The questions of representation of this data structure in memory, or strictness of evaluation, are not discussed.

Concrete Stream Calculus [29] introduces streams as codata. Streams are similar to McCarthy's definition of lists, in that they have functions head and tail, but they lack nil. This requires streams to be infinite structures only. The calculus is presented within Haskell, rendering all evaluation lazy.

Coinduction and codata are the usual way to introduce infinite data structures in programming languages [33, 39]. Key to the introduction of codata typically is the use of coinductive semantics [40]. In our paper, the use of ordinals keeps the semantics inductive and deals with infinite objects by means of ordinals. In [55], the author investigates a model of a total functional language, in which codata is used to define infinite data objects.

Streams are also related to dataflow models, such as [18, 37, 49]. The computation graphs in the latter can be seen as recursive expressions on potentially infinite streams. As demonstrated in [5], there is a demand to consider multi-dimensional infinite streams that cache their parts for better efficiency.

Two array representations, called push arrays and pull arrays, are presented in [51]. Pull arrays are treated as objects that have a length and an index-mapping function; push arrays are structures that keep sequences of element-wise updates. The imap defined here can be considered an advanced version of a pull array, with partitions and transfinite shape. The availability of partitions circumvents a number of inefficiencies, (e.g. embedded conditionals) of classical pull arrays; the ordinals, in the context of the imap-construct, enable the expression of streaming algorithms.

The \#Id language, presented in [27], is similar to $\lambda_{\omega}$; It combines the idea of lazy data structures with an eager execution context.

In [4, 45], the authors propose a system that makes it possible to reason whether a computation defined on an infinite stream is productive ${ }^{8}$ - a question that can be transferred directly to $\lambda_{\omega}$. Their technique lies in the introduction of a clock abstraction which limits the number of operations that can be made before a value must be returned. This approach has some analogies with defining explicit "windows" on arrays, as for example proposed in [25], or guarantees that programs run in constant space in [41].

One of the key features of the array language described in this paper is the availability of strict shape for any expression of the language. A similar effect can be achieved by encoding shapes in types. Specifically in the dependently-typed system, such an approach can be very powerful. The work on container theory [1] allows a very generic description of indexed objects capturing ideas of shapes and indices in types. A very similar idea in the context of arrays is described in [20]. The work on dependent type systems for array languages include [50, 54, 60]. Finally, a way to extend a type theory to include the notion of ordinals can be found in [26].

## 8 CONCLUSIONS AND FUTURE WORK

This paper proposes transfinite arrays as a basis for a simple applied $\lambda$-calculus $\lambda_{\omega}$. The distinctive feature of transfinite arrays is their ability to capture arrays with infinitely many elements, while maintaining structure within that infiniteness. The number of axes is preserved, and individual axes can contain infinitely many infinite subsequences of elements. This capability extends, into the transfinite space, many structural properties that hold for finite arrays.

The embedding of transfinite arrays into $\lambda_{\omega}$ allows for recursive array definitions, offering an opportunity to transliterate typical list-based algorithms, including algorithms on infinite lists for stream processing, into a generic array-based form. The paper presents several examples to this effect, and provides some efficiency considerations for them. It remains to be seen if these considerations, in practice, enable a truly unified view of arrays, lists, and streams.

The array-based setting of $\lambda_{\omega}$ allows this recursive style of defining infinite structures to be taken into a multi-dimensional context, enabling elegant specification of inherently multi-dimensional problems on infinite arrays. As an example, we present an implementation of Conway's Game of Life which, despite looking very similar to a formulation for finite arrays, is defined for positive infinities on both axes. Within $\lambda_{\omega}$, accessing neighbouring elements along both axes can be specified without requiring traversals of nested cons lists.

[^5]We also present an implementation for the Ackerman function, using a 2-dimensional transfinite array, one axis per parameter. The resulting code adheres closely to the abstract declarative formulation of the function, while also implicitly generating a basis for a memoising implementation of the algorithm.

An interesting aspect of transfinite arrays is that ordinal-based indexing opens up an avenue to express transfinite induction in data in very much the same way as nil and cons are duals to the principle of mathematical induction. This can not be done easily using cons lists, as there is no concept of limit ordinal in the list data structure. It may be possible to encode this principle by means of nesting, but then one would need a type system or some sort of annotations to distinguish lists of transfinite length from nested lists. The imap construct from the proposed formalism can be seen as an elegant solution to this.

The fact that imap supports random access and is powerful enough to capture list and stream expressions opens up an exciting perspective for the implementation of $\lambda_{\omega}$. When arrays are finite, it is possible to reuse one of the existing efficient array-based implementations. When arrays are infinite, we can use list or stream implementations to encode $\lambda_{\omega}$, but at the same time the properties of the original $\lambda_{\omega}$ programs open the door to rich program analysis. We believe that many functional languages striving for performance could benefit from the proposed design, at least when finiteness of arrays can be determined by program analysis.

Although the concept of transfinite arrays offers many new and interesting possibilities, we note several practical aspects that would benefit from further investigation. It is not yet clear what are the most efficient implementations for our proposed infinite structures. Choices of representation affect both memory management design and the guarantees that our semantics can provide. A type system for the proposed formalism is far from obvious, with the main question being the decidability of useful ordinal properties in a type system. The first-order theory of ordinal addition is known to be decidable [13], but more complex ordinal theories can quickly get undecidable [15]. To our knowledge, there is no type system that natively supports the notion of ordinals.

Furthermore, a number of extensions to the proposed formalism are possible. For example, it would be very useful to support streams that can terminate. Currently the only way one could model this in the proposed formalism is by introducing a new "end of stream" value, and defining an infinite stream, where from a certain index, all further values will be "end of stream".

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[^0]:    ${ }^{1}$ The implementation is freely available at https://github.com/ashinkarov/heh.

[^1]:    ${ }^{2}$ For readers familiar with Haskell: the imap defined here derives the index space from a shape expression. It does not require an argument array of that shape.
    ${ }^{3}$ A formal definition of the extended operator is: $(\dot{\oplus}) \equiv \lambda a . \lambda b . i m a p|a|\left\{\_(i v): a . i v \oplus b . i v\right.$ where $\oplus \in\{+,-, \cdots\}$.

[^2]:    ${ }^{4}$ Technically, we support ordinal values only up to $\omega^{\omega}$, as ordinals are constructed using the constant $\omega$ and $+,-, *, /$ and \% operations (no built-in ordinal exponentiation).
    ${ }^{5}$ Eq. 2 can be generalised and shown to hold in the multi-dimensional case, provided that ++ and drop operate over the same axis.

[^3]:    ${ }^{6}$ See this video by John Scholes for more details: https://youtu.be/a9xAKttWgP4

[^4]:    ${ }^{7}$ See http://www.sac-home.org for further details.

[^5]:    ${ }^{8}$ The computation will eventually produce the next item, i.e. it is not stuck.

